References:

(1) Shanhe Wu and Ovidiu Furdui, *A note on a conjectured Nesbitt type inequality*, Taiwanese Journal of Mathematics, 15 (2) (2011), 449-456.

Solution 3 by Soumitra Mandal, Chandar Nagore, India

$$\sum_{cyc} \frac{\log_a b + \log_b c}{m + n \log_a c} = \sum_{cyc} \frac{\log b + \frac{\log a \cdot \log c}{\log b}}{m \log a + n \log c}$$

$$= \sum_{cyc} \frac{\log b}{m \log a + n \log c} + \sum_{cyc} \frac{\frac{\log a \cdot \log c}{\log b}}{m \log a + n \log c}$$

$$= \sum_{cyc} \frac{(\log b)^2}{n \log a \cdot \log b + n \log c \cdot \log b} + \sum_{cyc} \frac{\left(\frac{1}{\log b}\right)^2}{\frac{m}{\log b \cdot \log c} + \frac{n}{\log b \cdot \log a}}$$

$$\overbrace{\geq}^{BERGSTROM} \frac{(\log a + \log b + \log c)^2}{(m+n)(\log a \cdot \log b + \log b \cdot \log c + \log c \cdot \log a)} +$$

$$+ \frac{\left(\frac{1}{\log a} + \frac{1}{\log b} + \frac{1}{\log c}\right)^2}{(m+n)\left(\frac{1}{\log a \cdot \log b} + \frac{1}{\log b \cdot \log c} + \frac{1}{\log c \cdot \log a}\right)} \geq \frac{3}{m+n} + \frac{3}{m+n} = \frac{6}{m+n}$$

Editor's Comments: **Anna V. Tomova of Varna, Bulgaria** approached the solution as follows: She showed that the left hand side of the inequality can be put into the canonical form of $X + Y + \dfrac{1}{XY}$. She then showed that this canonical form has a global minimum at $(1, 1)$, forcing it to have a minimal value of 3, and working with this she produced the final result.

Bruno Salgueiro Fanego of Viveiro, Spain noted that the stated problem is a specific case of a more general result. Namely: If $x, y, z \in (0, \infty)$ and $xyz = 1$, then

$$\frac{x+y}{m + \dfrac{n}{z}} + \frac{y+z}{m + \dfrac{n}{x}} + \frac{z+x}{m + \dfrac{n}{y}} \geq \frac{6}{m+n}.$$

He proved the more general result, and applied it to the specific case.

Also solved by Arkady Alt, San Jose, CA; Bruno Salgueiro Fanego of Viveiro, Spain; Ed Gray of Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Shravan Sridhar, Udupi, India; Albert Stadler, Herrliberg, Switzerland; Anna V. Tomova of Varna, Bulgaria, and the proposer.

5454: *Proposed by Arkady Alt, San Jose, CA*

Prove that for integers k and l, and for any $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, the following inequality holds:

$$k^2 \tan \alpha + l^2 \tan \beta \geq \frac{2kl}{\sin(\alpha + \beta)} - \left(k^2 + l^2\right) \cot(\alpha + \beta).$$

Solution 1 by Ed Gray, Highland Beach, FL

We rewrite the inequality by transposing

1) $k^2\left(\frac{\sin a}{\cos a}+\frac{\cos(a+b)}{\sin(a+b)}\right)+t^2\left(\frac{\sin b}{\cos b}+\frac{\cos(a+b)}{\sin(a+b)}\right)\geq\frac{2kt}{\sin(a+b)}$

Multiplying by $\sin(a+b)$

2) $k^2\left(\frac{\sin a(\sin(a+b))}{\cos a}+\cos(a+b)\right)+t^2\left(\frac{\sin b\sin(a+b)}{\cos b}+\cos(a+b)\right)\geq 2kt$

3) $k^2\left(\frac{\sin a\sin(a+b)+\cos a\cos(a+b)}{\cos a}\right)+t^2\left(\frac{\sin b\sin(a+b)+\cos b\cos(a+b)}{\cos b}\right)\geq 2kt$

4) $k^2\left(\frac{\cos b}{\cos a}\right)+t^2\left(\frac{\cos a}{\cos b}\right)\geq 2kt$

5) $\frac{k^2\cos^2 b+t^2\cos^2 a}{\cos a\cos b}\geq 2kt$

6) $k^2\cos^2 b+t^2\cos^2 a\geq 2kt\cos a\cos b$, and transposing,

7) $(k\cos b-t\cos a)^2\geq 0.$.

So we retrace our steps to obtain the original inequality.

Solution 2 by Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC

First we consider the case when $\alpha+\beta=\frac{\pi}{2}$, then $\sin(\alpha+\beta)=1$, $\cot(\alpha+\beta)=0$, and $\tan\beta=\cot\alpha$. From these we have

$$k^2\tan\alpha+t^2\tan\beta-\frac{2kl}{\sin(\alpha+\beta)}+\left(k^2+l^2\right)\cot(\alpha+\beta)=k^2\tan\alpha+l^2\cot\alpha-2lk=\left(k\sqrt{\tan\alpha}-l\sqrt{\cot\alpha}\right)^2\geq 0,$$

which completes the proof when $\alpha+\beta=\frac{\pi}{2}$.

Now suppose that $\alpha+\beta\neq\frac{\pi}{2}$. By using the identity $\cot(\alpha+\beta)=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$, we have

$$k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\cot(\alpha+\beta)-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad k^2\tan\alpha+l^2\tan\beta+(k^2+l^2)\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad \frac{k^2\tan^2\alpha+k^2\tan\alpha\tan\beta+l^2\tan\beta\tan\alpha+l^2\tan^2\beta+(k^2+l^2)-\left(k^2+l^2\right)\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad \frac{k^2\tan^2\alpha+l^2\tan^2\beta+(k^2+l^2)}{\tan\alpha+\tan\beta}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$=\quad \frac{k^2(1+\tan^2\alpha)+l^2(1+\tan^2\beta)}{\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}}-\frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{\dfrac{k^2\sec^2\alpha + l^2\sec^2\beta}{\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}}}{} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{k^2\dfrac{\cos\beta}{\cos\alpha} + l^2\dfrac{\cos\alpha}{\cos\beta}}{\sin(\alpha+\beta)} - \frac{2kl}{\sin(\alpha+\beta)}$$

$$= \frac{\left(\sqrt{k\dfrac{\cos\beta}{\cos\alpha}} - l\sqrt{k\dfrac{\cos\alpha}{\cos\beta}}\right)^2}{\sin(\alpha+\beta)} \geq 0.$$

Editor's Note: Most of the solvers mentioned that the inequality holds for all real values of k and l. **David Stone and John Hawkins of Georgia Southern University** when a bit further. They stated: "the conditions that α and β be first quadrant angles is an easy way to make $\sin(\alpha+\beta) \neq 0$ and $\tan\alpha, \tan\beta, \cot(\alpha+\beta)$ be defined and guarantee that $\cos\alpha\cos\beta\sin(\alpha+\beta) > 0$." But the proof shows that the inequality would be true for any values of α and β which satisfy these conditions.

Also solved by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; Boris Rays, Brooklyn, NY; Daniel Sitaru, "Theodor Costescu" National Economic College, Severin Mehedinti; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA; Anna V. Tomova, Varna, Bulgaria, and the proposer.

5455: *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Find all real solutions to the following system of equations:

$$\begin{aligned} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} &= \frac{1}{abc} \\ a + b + c &= abc + \frac{8}{27}(a+b+c)^3 \end{aligned}$$

Solution 1 by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND

Suppose a, b, c are real numbers satisfying our system. Consider the polynomial

$$\begin{aligned} g(x) &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \end{aligned}$$

The first equation of our original system implies $ab + ac + bc = 1$. So

$$g(x) = x^3 - \lambda x^2 + x - \mu$$

where $\lambda = a + b + c$ and $\mu = abc$. Note that the second equation of our original system can be written as $\lambda = \mu + \frac{8}{27}\lambda^3$. We make the usual substitution to get a depressed cubic: